Assignment 7

This homework is due *Thursday* Oct 23.

There are total 27 points in this assignment. 22 points is considered 100%. If you go over 22 points, you will get over 100% for this homework and it will count towards your course grade (up to 115%).

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper *and give credit to your collaborators in your pledge*. Your solutions should contain full proofs. Bare answers will not earn you much.

This assignment covers sections 3.5, 4.1 in Bartle–Sherbert.

(1) [2pt] (Exercise 3.5.4) Show directly from definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.

(2) [3pt] (3.5.9) If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence. (*Hint:* $x_{n+2} - x_n = (x_{n+2} - x_{n+1}) + (x_{n+1} - x_n)$. Generalize this to $x_{n+m} - x_n$.)

(3) [2pt] Let $X = (x_n)$ be a sequence in $\mathbb{R}$. Is it true that if for any $\varepsilon > 0$, there is a natural number $H = H(\varepsilon)$ such that for all $n > H$, $|x_n - x_{n+1}| < \varepsilon$, then X is a Cauchy sequence? (*Hint*: inspect partial sums of harmonic series, or look at $x_n = \sqrt{n}$).

(4) [3pt] (Theorem 4.1.2) Let $A \subseteq \mathbb{R}$. Prove that
  (a) If a number $c \in \mathbb{R}$ is a cluster point of A, then there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.
  (b) If there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$, then c is a cluster point of A.

(5) (Modified 4.1.1) In each case below, find a number $\delta > 0$ such that the corresponding inequality holds for all x such that $0 < |x - c| < \delta$. Give a *specific number* as your answer, for example $\delta = 0.0001$, or $\delta = 2.5$, or $\delta = 3/14348$, etc. (Not necessarily the largest possible.)
  (a) [1pt] $|x^3 - 1| < 1/2$, $c = 1$. (*Hint:* $x^3 - 1 = (x - 1)(x^2 + x + 1)$.)
  (b) [1pt] $|x^3 - 1| < 10^{-3}$, $c = 1$.
  (c) [1pt] $|x^3 - 1| < \frac{1}{10^{-3}}$, $c = 1$.
  (d) [2pt] $|x^2 \sin x^3 - 0| < 0.00001$, $c = 0$.

— see next page —

(6) Reminder. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f has limit $L \in \mathbb{R}$ at c if
$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.

Below you can find (erroneous!) "definitions" of a limit of a function. In each case describe, exactly which functions "have limit L at c" according to that "definition".

(a) [2pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
$\forall \varepsilon > 0\ \forall \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.

(b) [2pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
$\exists \varepsilon > 0\ \exists \delta > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.

(c) [2pt] Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c be a cluster point of A. We say that f "has limit $L \in \mathbb{R}$ at c" if
$\exists \delta > 0\ \forall \varepsilon > 0\ \forall x \in A,\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.

(7) [3pt] (Modified 4.1.9) *Use ε–δ definition of limit* to show that

(a) $\lim\limits_{x \to 2} \frac{1}{1-x} = -1$,

(b) $\lim\limits_{x \to 1} \frac{x}{1+x} = \frac{1}{2}$.

(8) [3pt] (4.1.11) Show that the following limits do not exist:

(a) $\lim\limits_{x \to 0} (x + \operatorname{sgn} x)$,

(b) $\lim\limits_{x \to 0} \sin(1/x^2)$.